

ON SOME NEW INTEGRAL INEQUALITIES FOR K_s^2

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ABSTRACT. In this paper we establish some new inequalities of Hadamard-type for product of convex and s -convex functions in the second sense.

1. INTRODUCTION

A largely applied inequality for convex functions, due to its geometrical significance, is Hadamard's inequality (see [2], [3] or [6]) which has generated a wide range of directions for extension and a rich mathematical literature. The following definitions are well known in the mathematical literature: a function $f : I \rightarrow \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad (1.1)$$

holds for all $x, y \in I$ and $t \in [0, 1]$. Geometrically, this means that if P, Q and R are three distinct points on the graph of f with Q between P and R , then Q is on or below chord PR .

In the paper [4] Hudzik and Maligranda considered, among others, the class of functions which are s -convex in the second sense. This class is defined in the following way: [1] A function $f : [0, \infty) \rightarrow \mathbb{R}$ is said to be s -convex in the second sense if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \quad (1.2)$$

holds for all $x, y \in [0, \infty)$, $t \in [0, 1]$ and for some fixed $s \in (0, 1]$. The class of s -convex functions in the second sense is usually denoted with K_s^2 .

It can be easily seen that for $s = 1$, s -convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

In the same paper [4] Hudzik and Maligranda proved that if $s \in (0, 1)$, $f \in K_s^2$ implies $f([0, \infty)) \subseteq [0, \infty)$, i.e., they proved that all functions from K_s^2 , $s \in (0, 1)$, are nonnegative.

Example 1. [4]. Let $s \in (0, 1)$ and $a, b, c \in \mathbb{R}$. We define function $f : [0, \infty) \rightarrow \mathbb{R}$ as

$$f(t) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0. \end{cases} \quad (1.3)$$

It can be easily checked that

- (1) If $b \geq 0$ and $0 \leq c \leq a$, then $f \in K_s^2$
- (2) If $b > 0$ and $c < 0$, then $f \notin K_s^2$

Many important inequalities are established for the class of convex functions, but one of the most famous is so called Hermite-Hadamard inequality (or Hadamard's inequality). This double inequality is stated as follows (see for example [7, p.137]): let f be a convex function on $[a, b] \subset \mathbb{R}$, where $a \neq b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (1.4)$$

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In the paper [8] Tunç established one new Hadamard-type inequality for products of convex functions. It is given in the next theorem.

Theorem 1. [8] Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two convex functions and $fg \in L^1([a, b])$. Then,

$$\begin{aligned} & \frac{1}{(b-a)^2} \int_a^b (b-x) (f(a)g(x) + g(a)f(x)) dx \\ & + \frac{1}{(b-a)^2} \int_a^b (x-a) (f(b)g(x) + g(b)f(x)) dx \\ & \leq \frac{1}{b-a} \int_a^b f(x)g(x) dx + \frac{M(a,b)}{3} + \frac{N(a,b)}{6}, \end{aligned} \quad (1.5)$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$, $N(a, b) = f(a)g(b) + f(b)g(a)$.

The main purpose of this paper is to establish new inequalities as given in Theorem 1, but now for the class of s -convex functions in the second sense by using the elementary inequalities.

2. MAIN RESULTS

In the our next theorems we will also make use of Beta function of Euler type, which is for $u, v > 0$ defined as

$$\beta(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$$

and

$$\beta(u, v) = \beta(v, u),$$

where the gamma function, denoted by $\Gamma(x)$, provides a generalization of factorial n to the case in which n is not an integer.

Theorem 2. Let $f, g : I \rightarrow \mathbb{R}$, $I \subset [0, \infty)$, $a, b \in I$, with $a < b$ be functions such that f, g and fg are in $L^1([a, b])$. f is convex and g is s -convex function in the second sense on $[a, b]$, for some $s \in (0, 1]$, then

$$\begin{aligned} & \frac{f(a)}{(b-a)^2} \int_a^b (b-x) g(x) dx + \frac{f(b)}{(b-a)^2} \int_a^b (x-a) g(x) dx \\ & + \frac{g(a)}{(b-a)^{s+1}} \int_a^b (b-x)^s f(x) dx + \frac{g(b)}{(b-a)^{s+1}} \int_a^b (x-a)^s f(x) dx \\ & \leq \frac{1}{b-a} \int_a^b f(x)g(x) dx + \frac{M(a,b)}{s+2} + \frac{N(a,b)}{(s+1)(s+2)} \end{aligned} \quad (2.1)$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

Proof. Since f is convex and g is s -convex on $[a, b]$, we have

$$\begin{aligned} f(ta + (1-t)b) & \leq tf(a) + (1-t)f(b) \\ g(ta + (1-t)b) & \leq t^s g(a) + (1-t)^s g(b) \end{aligned}$$

for all $t \in [0, 1]$. Now, using the elementary inequality [5, p.4] $(a - b)(c - d) \geq 0$ ($a, b, c, d \in \mathbb{R}$ and $a < b, c < d$), we get inequality:

$$\begin{aligned}
 & tf(a)g(ta + (1-t)b) + (1-t)f(b)g(ta + (1-t)b) \\
 & + t^s g(a)f(ta + (1-t)b) + (1-t)^s g(b)f(ta + (1-t)b) \\
 \leq & f(ta + (1-t)b)g(ta + (1-t)b) + t^{s+1}f(a)g(a) \\
 & + t(1-t)^s f(a)g(b) + t^s(1-t)f(b)g(a) \\
 & + (1-t)^{s+1}f(b)g(b)
 \end{aligned}$$

Integrating this inequality over t on $[0, 1]$, we deduce that

$$\begin{aligned}
 & f(a) \int_0^1 tg(ta + (1-t)b) dt + f(b) \int_0^1 (1-t)g(ta + (1-t)b) dt \\
 & + g(a) \int_0^1 t^s f(ta + (1-t)b) dt + g(b) \int_0^1 (1-t)^s f(ta + (1-t)b) dt \\
 \leq & \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b) dt \\
 & + f(a)g(a) \int_0^1 t^{s+1} dt + f(a)g(b) \int_0^1 t(1-t)^s dt \\
 & + f(b)g(a) \int_0^1 t^s(1-t) dt + f(b)g(b) \int_0^1 (1-t)^{s+1} dt
 \end{aligned}$$

By substituting $ta + (1-t)b = x$, $(a-b)dt = dx$, we obtain

$$\begin{aligned}
& f(a) \int_0^1 tg(ta + (1-t)b) dt + f(b) \int_0^1 (1-t)g(ta + (1-t)b) dt \\
& + g(a) \int_0^1 t^s f(ta + (1-t)b) dt + g(b) \int_0^1 (1-t)^s f(ta + (1-t)b) dt \\
& = \frac{f(a)}{(b-a)^2} \int_a^b (b-x)g(x) dx + \frac{f(b)}{(b-a)^2} \int_a^b (x-a)g(x) dx \\
& + \frac{g(a)}{(b-a)^{s+1}} \int_a^b (b-x)^s f(x) dx + \frac{g(b)}{(b-a)^{s+1}} \int_a^b (x-a)^s f(x) dx \\
& \leq \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b) dt \\
& + f(a)g(a) \int_0^1 t^{s+1} dt + f(a)g(b) \int_0^1 t(1-t)^s dt \\
& + f(b)g(a) \int_0^1 t^s(1-t) dt + f(b)g(b) \int_0^1 (1-t)^{s+1} dt \\
& = \frac{1}{b-a} \int_a^b f(x)g(x) dx + \frac{f(a)g(a) + f(b)g(b)}{s+2} \\
& + f(a)g(b)\beta(2, s+1) + f(b)g(a)\beta(s+1, 2) \\
& = \frac{1}{b-a} \int_a^b f(x)g(x) dx + \frac{M(a, b)}{s+2} \\
& + f(a)g(b)\beta(2, s+1) + f(b)g(a)\beta(2, s+1) \\
& = \frac{1}{b-a} \int_a^b f(x)g(x) dx + \frac{M(a, b)}{s+2} + \beta(2, s+1)[f(a)g(b) + f(b)g(a)] \\
& = \frac{1}{b-a} \int_a^b f(x)g(x) dx + \frac{M(a, b)}{s+2} + \frac{\Gamma(2)\Gamma(s+1)}{\Gamma(s+3)}N(a, b) \\
& = \frac{1}{b-a} \int_a^b f(x)g(x) dx + \frac{M(a, b)}{s+2} + \frac{\Gamma(s+1)}{\Gamma(s+3)}N(a, b) \\
& = \frac{1}{b-a} \int_a^b f(x)g(x) dx + \frac{M(a, b)}{s+2} + \frac{N(a, b)}{(s+1)(s+2)}
\end{aligned}$$

which completes the proof. \square

Remark 1. In Theorem 2, if we choose $s = 1$, then (2.1) reduces to (1.5)

Theorem 3. Let $f, g : I \rightarrow \mathbb{R}$, $I \subset [0, \infty)$, $a, b \in I$, $a < b$ be functions such that f, g and fg are in $L^1([a, b])$. If f is s_1 -convex and g is s_2 -convex in the second sense on $[a, b]$ for some $s_1, s_2 \in (0, 1]$, then

$$\begin{aligned}
& \frac{f(a)}{(b-a)^{s_1+1}} \int_a^b (b-x)^{s_1} g(x) dx + \frac{f(b)}{(b-a)^{s_1+1}} \int_a^b (x-a)^{s_1} g(x) dx \\
& + \frac{g(a)}{(b-a)^{s_2+1}} \int_a^b (b-x)^{s_2} f(x) dx + \frac{g(b)}{(b-a)^{s_2+1}} \int_a^b (x-a)^{s_2} f(x) dx \quad (2.2) \\
& \leq \frac{1}{b-a} \int_a^b f(x)g(x) dx + \frac{1}{s_1 + s_2 + 1} \left[M(a, b) + s_1 s_2 \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1 + s_2 + 1)} N(a, b) \right],
\end{aligned}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

Proof. Since f is s_1 -convex and g is s_2 -convex on $[a, b]$, we have

$$\begin{aligned} f(ta + (1-t)b) &\leq t^{s_1} f(a) + (1-t)^{s_1} f(b) \\ g(ta + (1-t)b) &\leq t^{s_2} g(a) + (1-t)^{s_2} g(b) \end{aligned}$$

for all $a, b \in I$ and $t \in [0, 1]$. Now, using the elementary inequality [5, p.4] $(a-b)(c-d) \geq 0$ ($a, b, c, d \in \mathbb{R}$ and $a < b, c < d$), we get inequality:

$$\begin{aligned} &t^{s_1} f(a) g(ta + (1-t)b) + (1-t)^{s_1} f(b) g(ta + (1-t)b) \\ &+ t^{s_2} g(a) f(ta + (1-t)b) + (1-t)^{s_2} g(b) f(ta + (1-t)b) \\ \leq &f(ta + (1-t)b) g(ta + (1-t)b) + t^{s_1+s_2} f(a) g(a) \\ &+ t^{s_1} (1-t)^{s_2} f(a) g(b) + t^{s_2} (1-t)^{s_1} f(b) g(a) \\ &+ (1-t)^{s_1+s_2} f(b) g(b) \end{aligned}$$

Integrating both sides of the above inequality over $[0, 1]$, we deduce that:

$$\begin{aligned} &f(a) \int_0^1 t^{s_1} g(ta + (1-t)b) dt + f(b) \int_0^1 (1-t)^{s_1} g(ta + (1-t)b) dt \\ &+ g(a) \int_0^1 t^{s_2} f(ta + (1-t)b) dt + g(b) \int_0^1 (1-t)^{s_2} f(ta + (1-t)b) dt \\ \leq &\int_0^1 f(ta + (1-t)b) g(ta + (1-t)b) dt \\ &+ f(a) g(a) \int_0^1 t^{s_1+s_2} dt + f(a) g(b) \int_0^1 t^{s_1} (1-t)^{s_2} dt \\ &+ f(b) g(a) \int_0^1 t^{s_2} (1-t)^{s_1} dt + f(b) g(b) \int_0^1 (1-t)^{s_1+s_2} dt \end{aligned}$$

By substituting $ta + (1 - t)b = x$, $(a - b)dt = dx$, we obtain

$$\begin{aligned}
& f(a) \int_0^1 t^{s_1} g(ta + (1 - t)b) dt + f(b) \int_0^1 (1 - t)^{s_1} g(ta + (1 - t)b) dt \\
& + g(a) \int_0^1 t^{s_2} f(ta + (1 - t)b) dt + g(b) \int_0^1 (1 - t)^{s_2} f(ta + (1 - t)b) dt \\
= & \frac{f(a)}{(b - a)^{s_1 + 1}} \int_a^b (b - x)^{s_1} g(x) dx + \frac{f(b)}{(b - a)^{s_1 + 1}} \int_a^b (x - a)^{s_1} g(x) dx \\
& + \frac{g(a)}{(b - a)^{s_2 + 1}} \int_a^b (b - x)^{s_2} f(x) dx + \frac{g(b)}{(b - a)^{s_2 + 1}} \int_a^b (x - a)^{s_2} f(x) dx \\
\leq & \int_0^1 f(ta + (1 - t)b) g(ta + (1 - t)b) dt \\
& + f(a) g(a) \int_0^1 t^{s_1 + s_2} dt + f(a) g(b) \int_0^1 t^{s_1} (1 - t)^{s_2} dt \\
& + f(b) g(a) \int_0^1 t^{s_2} (1 - t)^{s_1} dt + f(b) g(b) \int_0^1 (1 - t)^{s_1 + s_2} dt \\
= & \frac{1}{b - a} \int_a^b f(x) g(x) dx + \frac{f(a)g(a) + f(b)g(b)}{s_1 + s_2 + 1} \\
& + f(a)g(b)\beta(s_1 + 1, s_2 + 1) + f(b)g(a)\beta(s_2 + 1, s_1 + 1) \\
= & \frac{1}{b - a} \int_a^b f(x) g(x) dx + \frac{M(a, b)}{s_1 + s_2 + 1} \\
& + f(a)g(b)\beta(s_1 + 1, s_2 + 1) + f(b)g(a)\beta(s_1 + 1, s_2 + 1) \\
= & \frac{1}{b - a} \int_a^b f(x) g(x) dx + \frac{M(a, b)}{s_1 + s_2 + 1} \\
& + \beta(s_1 + 1, s_2 + 1) [f(a)g(b) + f(b)g(a)] \\
= & \frac{1}{b - a} \int_a^b f(x) g(x) dx + \frac{M(a, b)}{s_1 + s_2 + 1} + \frac{\Gamma(s_1 + 1)\Gamma(s_2 + 1)}{\Gamma(s_1 + s_2 + 2)} N(a, b) \\
= & \frac{1}{b - a} \int_a^b f(x) g(x) dx + \frac{1}{s_1 + s_2 + 1} \left[M(a, b) + s_1 s_2 \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1 + s_2 + 1)} N(a, b) \right]
\end{aligned}$$

which completes the proof. \square

Remark 2. In Theorem 3, if we choose $s_1 = s_2 = 1$, then (2.2) reduces to (1.5)

Corollary 1. With the above assumptions and under the conditions that $s_1 = s_2 = 1$ and $x = \frac{a+b}{2}$, the following the inequality will be obtained

$$\begin{aligned}
& \frac{f(a) + f(b)}{2} g\left(\frac{a+b}{2}\right) + \frac{g(a) + g(b)}{2} f\left(\frac{a+b}{2}\right) \\
\leq & f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + \frac{M(a, b)}{3} + \frac{N(a, b)}{6}.
\end{aligned} \tag{2.3}$$

Remark 3. Similarly to Hadamard's inequality applications, some applications to special means can be deduced by the above obtained two new theorems.

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